Accounting for Spatial Error Correlations in Altimetric Data Assimilation

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Introduction

The Kalman filter is a widely spread data assimilation method in oceanography. The standard Kalman filter observational update requires the inversion of the innovation error covariance matrix, what is prohibitive regarding its size. Most implementations of the Ensemble Kalman filter circumvent this difficulty assuming the diagonality of the observation error covariance matrix, what makes the analysis calculation numerically tractable. However, when observation errors are actually correlated spatially, such hypothesis yields too much weight to the observations, and may lead to an inappropriate use of the observations. Spatial altimetric measurements, because they are performed along tracks, are very likely subject to spatial error correlations. In this presentation, we describe a parameterization of the observation error covariance matrix that preserves its diagonal shape, but represents a simple first order autoregressive correlation structure of the observation errors. This parameterization is based upon an augmentation of the observation vector with gradients of observations. Numerical applications to ocean altimetry show the detrimental effects of specifying the matrix diagonal when observations errors are correlated, and how the new parameterization not only removes the detrimental effects of correlations, but also makes use of these correlations to improve the data assimilation products. A detailed presentation is available in *Brankart et al.* (2009).

Analysis update in square root or ensemble Kalman filters

In Ensemble Square Root Filters (ESRF), the covariance matrix is decomposed as $\mathbf{P}^f = \mathbf{S}^f\mathbf{S}^{fT}$. The ESRF correction is either calculated with (using a serial processing of observations; *Houtekamer and Mitchell*, 2001)

$$\delta\mathbf{x} = \mathbf{S}^f(\mathbf{H}\mathbf{S}^f)^T\left[(\mathbf{H}\mathbf{S}^f)(\mathbf{H}\mathbf{S}^f)^T + \mathbf{R}\right]^{-1}(\mathbf{y} - \mathbf{H}\mathbf{x}^f), \tag{1}$$

or with a prior transformation featuring $\Gamma = (\mathbf{H}\mathbf{S}^f)^T\mathbf{R}^{-1}(\mathbf{H}\mathbf{S}^f)$ (*Pham et al.*, 1998),

$$\delta\mathbf{x} = \mathbf{S}^f\left[\mathbf{I} + \Gamma\right]^{-1}(\mathbf{H}\mathbf{S}^f)^T\mathbf{R}^{-1}(\mathbf{y} - \mathbf{H}\mathbf{x}^f). \tag{2}$$

In the first case, the serial processing implies that observation errors are uncorrelated ($\mathbf{R}$ is diagonal); In the second case, as the inverse of $\mathbf{R}$ is required, $\mathbf{R}$ is often considered diagonal for simplicity, even if observations are actually correlated.

Linear transformation of the observation vector to simulate correlations

Rationale

We consider a observation transformation matrix $\mathbf{T}$. Define

- $\mathbf{y}^+ = \mathbf{T}\mathbf{y}$, with covariance matrix $\mathbf{R}^+$;
- $\mathbf{H}^+ = \mathbf{T}\mathbf{H}$ the transformed observation operator.

By computing the new increment $\delta\mathbf{x}^+$ using Eq. 2, it can be shown that it is equivalent to assimilate $(\mathbf{y}^+, \mathbf{R}^+)$ and $(\mathbf{y}, \mathbf{R})$ if $\mathbf{y}^+ = \mathbf{T}\mathbf{y}$ and $\mathbf{R}^{-1} = \mathbf{T}^T\mathbf{R}^{+-1}\mathbf{T}$.

Example of application: gradient operator in one dimension

Let us introduce the transformation $\mathbf{y}^+ = \mathbf{T}\mathbf{y} = \begin{bmatrix}\mathbf{T}_1\\ \mathbf{T}_2\end{bmatrix}\mathbf{y}$ where $\mathbf{T}_1$ is the identity matrix, $\mathbf{T}_2$ the gradient operator, $\mathbf{T}_{2,ij} = \frac{\delta_{ij} - \delta_{i-1,j}}{\Delta\xi}$. If $\mathbf{R}^+$ is homogeneous, i.e. $\mathbf{R}^+ = \begin{bmatrix}\sigma_0^2\mathbf{I} & 0\\ 0 & \sigma_1^2\mathbf{I}\end{bmatrix}$, then $\mathbf{R}$ verifies

$$\mathbf{R}^{-1} = \frac{1}{\sigma_0^2}\mathbf{I} + \frac{1}{\sigma_1^2\Delta\xi^2}\begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 & 0\\ -1 & 2 & -1 & \ddots & & 0 & 0\\ 0 & -1 & 2 & \ddots & \ddots & & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\ 0 & & \ddots & \ddots & 2 & -1 & 0\\ 0 & 0 & & \ddots & -1 & 2 & -1\\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \tag{3}$$

and it can be proven that this is a consistent discretization of the inverse of the covariance function

$$\mathcal{R}(\rho) = \frac{\sigma_0^2}{2}\exp\left(-\frac{|\rho|}{\ell}\right) \quad \text{with} \quad \ell = \frac{\sigma_0}{\sigma_1} \tag{4}$$

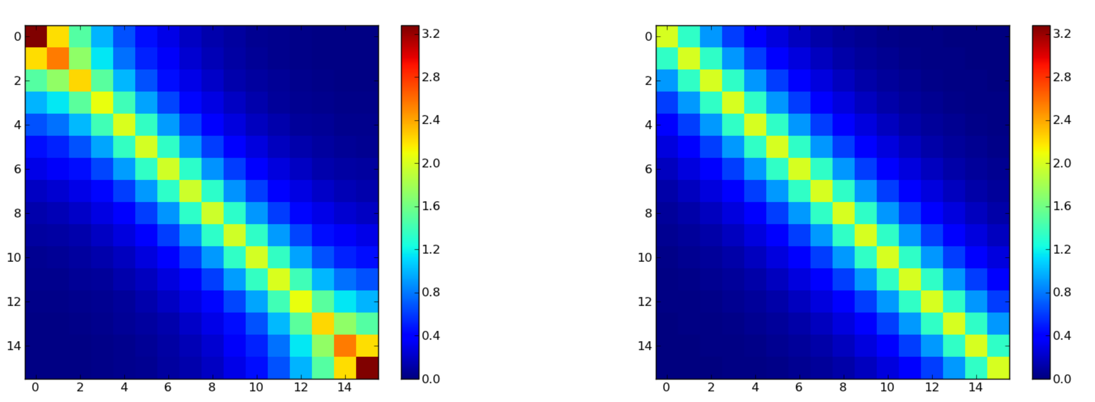


Figure 1: Covariance matrices obtained numerically (using Eq. 3) and analytically (Eq. 4) with $\sigma_0^2 = 4$, $\sigma_1^2 = 1$, $\Delta\xi = 0.8$.

Application to altimetry in the North Brazil current

Experiment: A 5-year simulation of the circulation in the North Brazil current region is performed with a regional configuration of the NEMO model. The 300 output snapshots (one every 6 days) determine the *true* states. The mean of this ensemble is taken as the *background* state (see Figure 2, upper panel). To parameterize the background error covariance matrix, we use the covariance of 59 snapshots (one per month over 5 years, except those that are less than 1 month away from the true state). Figure 2, bottom panel, illustrates what the square root of the matrix diagonal looks like.

As *observation*, Sea Surface Height (SSH) is observed over the full domain, with a 4 cm error standard deviation. **Two observation vectors** are generated from the true state: a first one, by adding uncorrelated observation noise, and a second one, by adding a correlated observation noise, with a covariance matrix similar to Eq. 3 for the 2-dimensional case. The noise is scaled to have a uniform standard deviation $\sigma = 0.04$ m. The observation error covariance is parameterized either with a diagonal matrix, or with a non-diagonal matrix, following the method described previously.

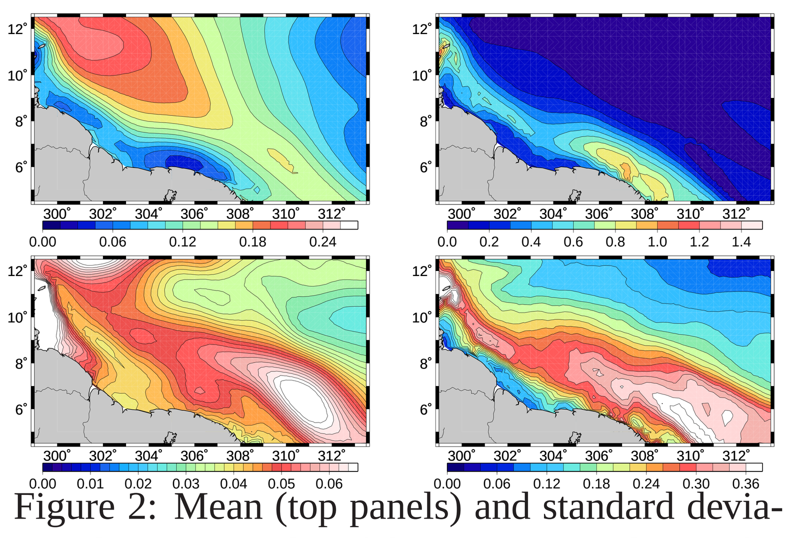
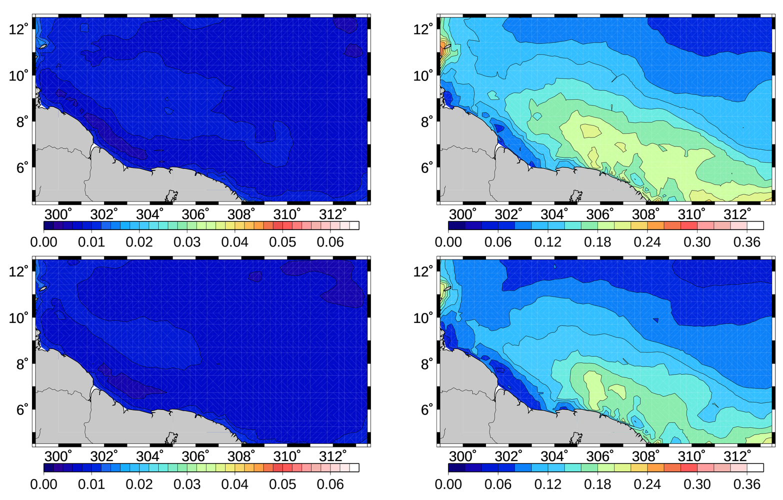


Figure 2: Mean (top panels) and standard deviation (bottom panels) of the 5 years simulation, for the sea surface height (in m, left panels), and sea surface velocity (in m/s, right panels).

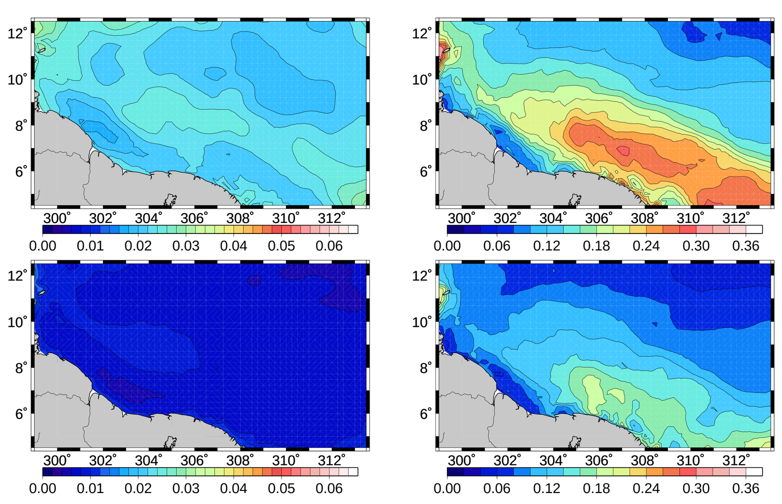
Uncorrelated errors



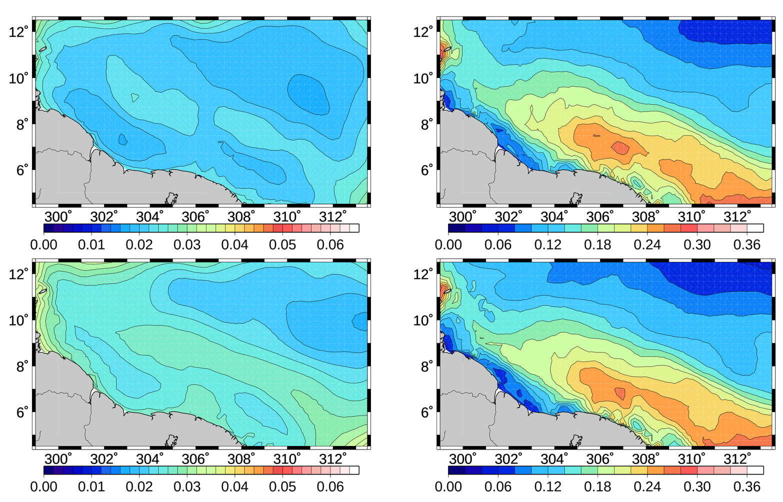
Observation errors are spatially uncorrelated, and $\mathbf{R}$ is diagonal. Figure 3 on the left shows the error standard deviation, as measured using the ensemble (top panels), and as estimated by the scheme (the square root of the diagonal of $\mathbf{P}^a$, bottom panels). Both are consistent for altimetry (in m, left panels) and for velocity (in m/s, right panels).

Correlated errors, with diagonal $\mathbf{R}$ parameterization

Observation errors are spatially correlated, but $\mathbf{R}$ is taken diagonal. Figure 4 on the right displays the same fields as Figure 3, for this experiment. The inappropriate parameterization of $\mathbf{R}$ leads to a significant discrepancy between the errors estimated by the ensemble and by the error modes.



Correlated errors, with consistent $\mathbf{R}$ parameterization



Observation errors are spatially correlated, and $\mathbf{R}$ is parameterized using the observation gradient method. The coherence between errors estimated with the ensemble and with the error modes is restored. Residual errors are higher than in the uncorrelated case for altimetry, but not much for velocity. Correlations in observations of altimetry actually provide quantitative information on velocity.

Conclusions

A new parameterization of the observation error covariance matrix has been proposed, which:

- is based on an augmented observation vector approach (with gradients of observations here);
- represents some types of spatial correlations between observation errors;
- preserves the numerical efficiency of Ensemble filters schemes.

This parameterization has been used to study the impact of SSH observation correlations on SSH and velocity analysis. It has been proven accurate and efficient.

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